NOTATION

$\mathbf{B}$ = $\mathbf{P}^T\mathbf{P} + \mathbf{I}$
$\mathbf{C}$ = $(r \times n - r)$ matrix, $(\partial g_i/\partial d_a)$, north-east portion of $\mathbf{G}$
$\mathbf{d}$ = $(n - r \times 1)$ vector of decision variables, (d_a)
e = solution to Equations (25) or (27)
f = objective function
F = Lagrange function
g = constraining function
$\mathbf{G}$ = $(m \times n)$ matrix of rank r, $(\partial g_l/\partial x_\alpha)$
$\mathbf{H}$ = $(n \times n)$ Hessian matrix $(\partial^2 f/\partial x_\alpha \partial x_\beta)$
$\mathbf{H}_c$ = $(n - r \times n - r)$ constrained Hessian matrix, $(\delta^2 f/\delta d_a \delta d_b)$
$\mathbf{H}_F$ = $(n \times n)$ matrix, $(\partial^2 F/\partial x_\alpha/\partial x_\beta)$
$\mathbf{J}$ = $(r \times r)$ matrix, leading portion of $\mathbf{G}$, $(\partial g_i/\partial s_j)$
m = number of constraints
n = number of independent variables
p = polynomial of degree $(n - r)$
$\mathbf{P}$ = $(r \times n - r)$ matrix, $\mathbf{J}^{-1}\mathbf{C}$
r = rank of $\mathbf{G}$ and $\mathbf{J}$
$\mathbf{s}$ = $(r \times 1)$ vector of state variables, (s_i)
$\mathbf{t}^a$ = $(r \times n - r)$ vector, a^{th} row of $\mathbf{T}$
$\mathbf{t}_b$ = $(n - r \times 1)$ vector, b^{th} column of $\mathbf{T}$
$\mathbf{T}$ = $(n - r \times n - r)$ matrix, defined by Equation (9)
$\mathbf{x}$ = $(n \times 1)$ vector of independent variables, (x_α)

Other Symbols

a, b = dummy indexes, $1 \leq a, b \leq (n - r)$
i, j, k = dummy indexes, $1 \leq i, j, k \leq r$
l = dummy index, $1 \leq l \leq m$
α, β = dummy indexes, $1 \leq \alpha, \beta \leq n$
λ = Lagrange multiplier
ϕ = quadratic form
∇_c = $(n - r \times 1)$ vector operator $(\delta/\delta d_a)$
∇_s = $(r \times 1)$ vector operator, $(\partial/\partial s_i)$

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On the Possibility of Stabilizing a Simple Negative Feedback Control System by Increasing Controller Gain on a PID Controller

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The purpose of this note is to show under what conditions a simple closed loop system may be stable at low values of K_c, unstable at moderate values, and be restabilized at higher values of K_c. Such an effect is possible with PID control but not with PI or PD control systems (Coughanowr and Koppel, 1965).

Consider the characteristic equation (using Laplace transforms) for a system with three first-order transfer functions and a PID controller:

$$\tau_1\tau_2\tau_3 S^3 + (\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3)S^2 + (\tau_1 + \tau_2 + \tau_3)S + 1 + K_c[1 + \tau_D S + 1/(\tau_i S)] = 0 \quad (1)$$

If we let $\gamma_1 = \tau_1\tau_2\tau_3$, $\gamma_2 = \tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3$, and $\gamma_3 = \tau_1 + \tau_2 + \tau_3$, and multiply Equation (1) out, we obtain

$$\gamma_1\tau_i S^4 + \gamma_2\tau_i S^3 + (\gamma_3 + K_c\tau_D)\tau_i S^2 + (K_c + 1)\tau_i S + K_c = 0 \quad (2)$$

The required values of K_c, τ_D, and τ_i that will insure stable operation can be determined using the Routh array (Coughanowr and Koppel, 1965). Since all the parameters are positive, the conditions for stability can be written as

$$\frac{\gamma_2(\gamma_3 + K_c\tau_D)\tau_i^2 - \gamma_1(K_c + 1)\tau_i^2}{\gamma_2\tau_i} > 0 \quad (3)$$

and

$$(K_c + 1)\tau_i - \frac{K_c\gamma_2^2\tau_i^2}{[\gamma_2(\gamma_3 + K_c\tau_D) - \gamma_1(K_c + 1)]\tau_i^2} > 0 \quad (4)$$

Equation (3) can be rewritten to show the dependence of K_c on τ_D:

$$K_c > \frac{\gamma_1/\gamma_2 - \gamma_3}{\tau_D - \gamma_1/\gamma_2} \quad \text{if} \quad \tau_D > \gamma_1/\gamma_2 \quad (5a)$$

$$K_c < \frac{\gamma_1/\gamma_2 - \gamma_3}{\tau_D - \gamma_1/\gamma_2} \quad \text{if} \quad \tau_D < \gamma_1/\gamma_2 \quad (5b)$$

It can be easily shown that $\gamma_3 > \gamma_1\gamma_2$ for all positive τ_1, τ_2, τ_3 [that is, $(\tau_1 + \tau_2 + \tau_3)\cdot(\tau_1\tau_2 + \tau_1\tau_3 + \tau_3\tau_2) > \tau_1\tau_2\tau_3$]. Consequently, any positive value of K_c will satisfy Equation (3) [or (5a)] if $\tau_D > \gamma_1\gamma_2$. For $\tau_D < \gamma_1\gamma_2$, some high values of K_c will not satisfy Equation (3) [or (5b)] and will lead to instabilities.

For a system to be stable, it must simultaneously

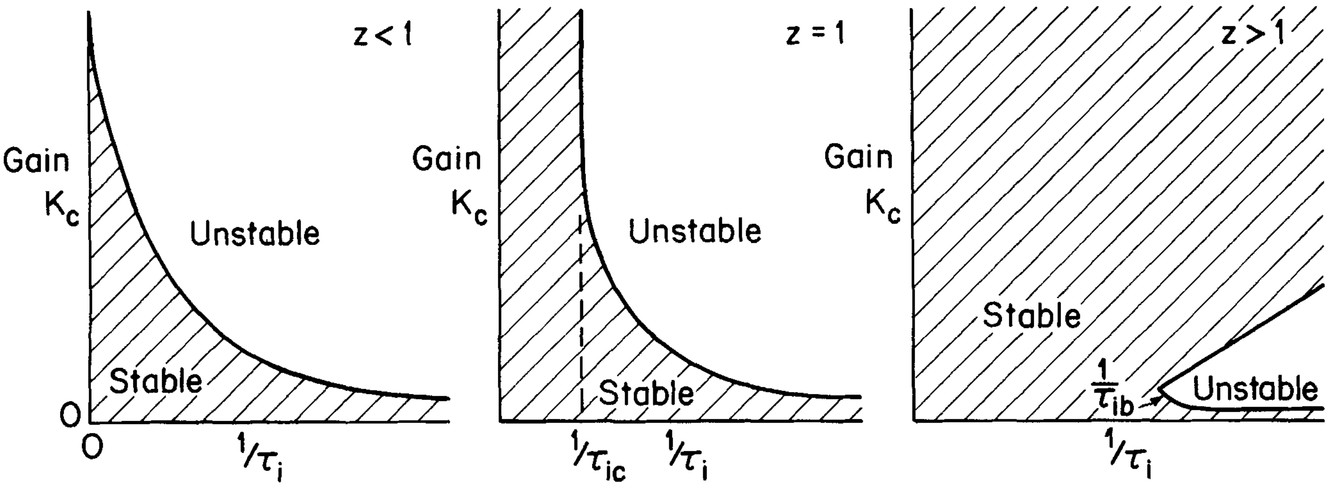

Fig. 1. The effect of reset rate on allowable gains for stable operation are sketched for various classes of values of z. For $z > 1$, a split stability region will always exist for $\tau_i < \tau_{ib}$ [Equation (10)] At $z = 1$, the system will be stable for any K_c for $1/\tau_i = (\gamma_3/\gamma_2 - \gamma_1/\gamma_2)$ denoted by $1/\tau_{ic}$ in this sketch.

satisfy Equations (3) and (4). If Equation (3) is satisfied, Equation (4) can be rewritten as

$$(K_c + 1)[\gamma_2(\gamma_3 + K_c\tau_D) - \gamma_1(K_c + 1)] - K_c\gamma_2^2/\tau_i > 0 \tag{6}$$

This can be rearranged to give

$$K_c^2 + [\alpha_1 + 1 - \alpha_2/\tau_i]K_c + \alpha_1 > 0 \tag{7a}$$

where

$$\alpha_1 = \frac{(\gamma_2/\gamma_1)\gamma_3 - 1}{z - 1} \tag{7b}$$

$$\alpha_2 = \frac{\gamma_2^2/\gamma_1}{z - 1} \tag{7c}$$

$$z = \tau_D\gamma_2/\gamma_1 \tag{7d}$$

Mathematically realizable values of K_c are

$$K_c = \tfrac{1}{2}[\alpha_2/\tau_i - (\alpha_1 + 1) \pm \sqrt{(\alpha_2/\tau_i - (\alpha_1 + 1))^2 - 4\alpha_1}] \tag{8}$$

If

$$(\alpha_1 + 1) > \alpha_2/\tau_i \tag{9}$$

then any value of positive K_c is allowable. However, if Equation (9) is not satisfied, it is possible to have two positive real roots for K_c if the following relationship is satisfied:

$$\tau_i < \frac{\alpha_2}{\alpha_1 + 1 + 2\sqrt{\alpha_1}} = \tau_{ib} \tag{10}$$

The system will have complex roots for $\alpha_2/(\alpha_1 + 1 + 2\sqrt{\alpha_1}) < \tau_i < \alpha_2/(\alpha_1 + 1 - 2\sqrt{\alpha_1})$ and one positive and one negative real root for $\tau_i > \alpha_2/(\alpha_1 + 1 - 2\sqrt{\alpha_1})$.

The expected variation in the stability of the system is sketched in Figure 1. For $z > 1$, it is possible with some values of τ_i [defined by Equation (10)] to have stable operation at low value of K_c; to enter the unstable region by increasing K_c; and, with further increases in K_c, to restabilize the system.

Generally, the values of τ_1, τ_2, τ_3, τ_D, and τ_i encountered in practice will be outside of this split stability region. However, some potentially realistic examples can be found. For example, if $\tau_1 = 1$, $\tau_2 = 30$, and $\tau_3 = 60$ min, then the Ziegler-Nichol's rules (Coughanowr and Koppel, 1965) would suggest $\tau_{iZ-N} = 14$ and $\tau_{DZ-N} = 3.5$ min. In this case, $\gamma_1 = 1\,800$ min^3, $\gamma_2 = 1\,890$ min^2, $\gamma_3 = 91$ min, and $z = 3.675$. Solving Equation (7a), we find that the system will be stable for $K_c < 2.50$ and for $K_c > 14.1$. In Figure 2, the stability plot for this system is given. If $\tau_1 = 1$, $\tau_2 = 20$, and $\tau_3 = 40$ min, the split stability region would fall outside of the values of τ_i and τ_D that would be selected by the Ziegler-Nichols rules. The split stability region becomes more and more remote from the recommended operating conditions as τ_2 and τ_3 are decreased in relation to τ_1.

This effect can be shown more quantitatively by proving that the ratio of the time constants (that is, τ_2/τ_1, τ_3/τ_1) controls the entrance of the Ziegler-Nichols settings into the split stability region. The Ziegler-Nichols settings are found by letting $\tau_{iZ-N} = \pi/\omega_c$ and $\tau_{DZ-N} = \pi/4\omega_c$, where ω_c is the frequency at which the system excluding the controller exhibits a 180 deg phase lag. For the system discussed in this note, ω_c is found from

$$\tan^{-1}\omega_c\tau_1 + \tan^{-1}\omega_c\tau_2 + \tan^{-1}\omega_c\tau_3 = -180 \text{ deg} \tag{11}$$

As an aside, it may be noted that although Equation (11) will give an unique value of ω_c, the Bode diagram drawn with the inclusion of the frequency response of the PID controller may give multiple values of ω_c.

Equation (11) can be solved for ω_c by twice making use of the identity (Weast and Selby, 1975)

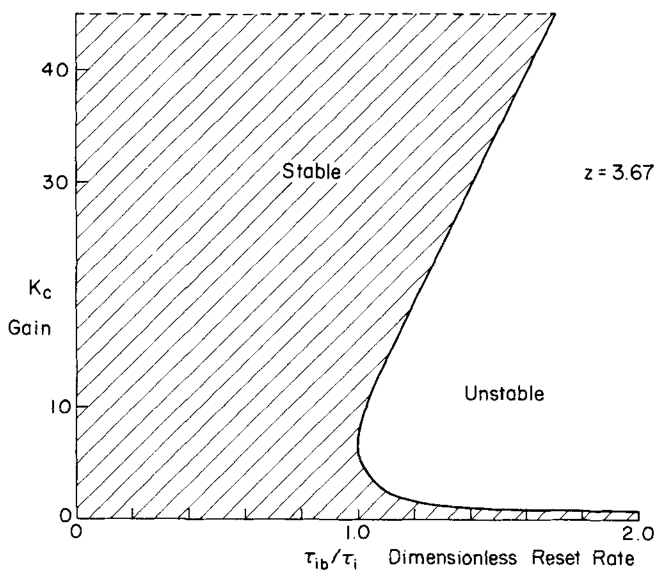

Fig. 2. This plot depicts the stability of a system with time constant ratios of $R_1 = 30$ and $R_2 = 60$ and a value of $z = 3.67$. The Ziegler-Nichol's rules would suggest that $\tau_{ib}/\tau_{iZN} = 1.10$; such a control system design would lead ideally to unstable operation for $250 < K_c < 14.1$.

$$\tan(A + D) = \frac{\tan A + \tan D}{1 - \tan A \tan D} \quad (12)$$

with the result that

$$\omega_c = [\gamma_3/\gamma_1]^{1/2} \quad (13)$$

If each time constant is multiplied by X, a positive constant, and Equations ($7d$) and (13) are combined with the definitions of γ_1, γ_2, γ_3, z, and τ_{DZ-N}, the following equations can be obtained:

$$z = \frac{\pi\gamma_2/\gamma_1}{4[\gamma_3/\gamma_1]^{1/2}} \quad (14a)$$

$$z_X = \frac{\pi X^2\gamma_2/X^3\gamma_1}{4[X\gamma_3/X^3\gamma_1]^{1/2}} \quad (14b)$$

Comparing Equations ($14a$) and ($14b$), it is evident that $z = z_X$ and that the value of z will depend only on the ratio of the time constants, not their absolute values. By a similar approach, it is easy to demonstrate that α_1 depends only on the ratio of the time constants. It can also be demonstrated that τ_{ib}/τ_{iZ-N} is dependent only on the time constant ratio. Using Equations ($7c$), (10), and the definition of τ_{iZ-N}, we can write

$$\left[\left(\frac{\gamma_2^2/\gamma_1}{z-1}\right) \Big/ (\alpha_1 + 1 + 2\sqrt{\alpha_1})\right] \Big/ \left(\frac{\pi}{\omega_c}\right) = \frac{\tau_{ib}}{\tau_{iZ-N}} \quad (15a)$$

$$\left[\left(\frac{X^4\gamma_2^2/X^3\gamma_1}{z-1}\right) \Big/ (\alpha_1 + 1 + 2\sqrt{\alpha_1})\right] \Big/ \left(\frac{\pi X}{\omega_c}\right) = \frac{\tau_{ib}}{\tau_{iZ-N}}\Bigg|_X \quad (15b)$$

Again, it is evident that Equation ($15b$) reduces to ($15a$).

The Ziegler-Nichols settings will enter the split stability regions if $\tau_{ib}/\tau_{iZ-N} > 1$. Since the ratio of the time constants controls the value of τ_{ib}/τ_{iZ-N}, it is clear that entrance into the split stability region depends only on the ratio of the time constants.

If $R_1 = \tau_2/\tau_1$, $R_2 = \tau_3/\tau_1$, and $R_2 = yR_1$, the critical value of R_1 which will cause the system to fall within the split stability region with τ_{iZN} and τ_{DZN} chosen by the Ziegler-Nichol's recommendations can be found. In Equation ($15a$), τ_{ib}/τ_{iZN} is set equal to unity, and the appropriate definitions for γ_1, γ_2, and γ_3 are recast in terms of R_1 and y. Once y is chosen, ω_c, z, α_1, and α_2 are functions of R_1 only since they are expressable solely in terms of γ_1, γ_2, and γ_3. These parameters are substituted into Equation ($15a$) which leads to an implicit expression for R_1. For example, if $y = 2$, R_1 critical will be 24.8.

The above analysis has been done with an ideal PID controller; the split stability region is due to the phase lead added by the controller at high frequencies. Real controllers have a maximum phase lead of about 60 deg as compared to the 90 deg phase lead of an ideal controller. Also, the phase lead in a real controller approaches 0 deg at high frequencies in contrast to the ideal control which maintains a phase lead of 90 deg as ω approached infinity. Consequently, it may be possible that the affect predicted using the ideal controller function is not realizable in practice. This possibility can be tested as follows.

First, some functional form describing controller performance must be found. Although several alternate forms are possible, one suggested by Harriott (1965) describing a pneumatic controller has been selected for this illustration and can be written as

$$G_c = K_c \frac{(1 + \tau_D S + 1/\tau_i S)}{(1 + \tau_D S/B + 1/\tau_i BS)} \quad (16)$$

Table 1. Changes in Allowable Values of K_c as a Function of B for Three First-Order Processes with Real PID Control. Numerical Results are for $\gamma_1 = 1\,800$, $\gamma_2 = 1\,890$, $\gamma_3 = 91$, $\tau_D = 3.5$, and $\tau_i = 14$ min

B	Values of K_c required for stability
50	$0 < K_c < 3.14$; $13.2 < K_c < 4\,617$
40	$0 < K_c < 3.34$; $12.9 < K_c < 3\,738$
30	$0 < K_c < 3.71$; $12.3 < K_c < 2\,857$
20	$0 < K_c < 4.71$; $10.9 < K_c < 1\,975$
15	$0 < K_c < 7.54$; $7.68 < K_c < 1\,531$
12	$0 < K_c < 1\,264$
10	$0 < K_c\ 1\,084$
5	$0 < K_c < 620$

Using this form, the characteristic equation becomes

$$\gamma_1 S^3 + \gamma_2 S^2 + \gamma_3 S + 1 + K_c \frac{(1 + \tau_D S + 1/\tau_i S)}{(1 + \tau_D S/B + 1/\tau_i BS)} = 0 \quad (17a)$$

or

$$(\gamma_1\tau_D\tau_i/B)S^5 + (\gamma_1\tau_i + \gamma_2\tau_D\tau_i/B)S^4 + (\gamma_1/B + \gamma_2\tau_i + \gamma_3\tau_D\tau_i/B)S^3 + (\gamma_3\tau_i + K_c\tau_D\tau_i + \gamma_2/B + \tau_D\tau_i/B)S^2 + [(K_c + 1)\tau_i + \gamma_3/B]S + K_c + 1/B = 0 \quad (17b)$$

Since the Routh array will ultimately include cubic terms, it is difficult to develop an analytical solution. The following numerical values were substituted to test the response of the system: $\gamma_1 = 1800$, $\gamma_2 = 1890$, $\gamma_3 = 91$, $\tau_D = 3.5$, and $\tau_i = 14$ (see Figure 2). Further, we will let B vary from 5 to 50 (see Buckley, 1964). The results from the Routh stability test are given in Table 1. The split stability region still exists for at least all B greater than 15.

For this system of three first-order transfer functions, it is apparent that some real controllers may exhibit a split stability region. Intuitively, the closer a controller approximates an ideal PID controller, the more likely such affects will exist. The addition of significant time delay to this particular system may eliminate the split stability region by cancelling the phase lead added by derivative action. Undoubtedly, split stability effects can exist for systems other than three first-order transfer functions, but no attempt has been made to discover these.

In this note, the conditions for entrance into a region of split stability with ideal PID control of three first-order processes have been found. Values of τ_i and τ_D chosen by the Ziegler-Nichols' rules can lead to operation in the split stability region, and such affects may be observable with some real controllers if time delay is insignificant.

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Helpful conversations with Mr. Ian Webster and Professor Peter Harriott are gratefully acknowledged.

NOTATION

- B = parameter to modify ideal controller action to approximate real controller
- K_c = controller gain
- R_1 = ratio of second largest to smallest time constant
- R_2 = ratio of largest to smallest time constant
- S = Laplace transform parameter

y = ratio of R_2 to R_1

z = ratio of the derivative time constant to γ_1/γ_2; see Equation (7*d*)

X = positive constant used to multiply each process time constant in a set

Greek Letters

α_1 = defined by Equation (7*b*)

α_2 = defined by Equation (7*c*), s

γ_1 = product of the process time constants; $\tau_1 \cdot \tau_2 \cdot \tau_3$, s

γ_2 = $\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3$, s^2

γ_3 = sum of process time constants; $\tau_1 + \tau_2 + \tau_3$, s

τ_1 = smallest process time constant, s

τ_2 = second largest process time constant, s

τ_3 = largest process time constant, s

τ_i = reset or integral action time constant, s

τ_D = derivative action time constant, s

τ_{iZ-N}; τ_{DZ-N} = values of τ_i and τ_D suggested by the Ziegler-Nichol's rules, s

τ_{ib} = all lower values of τ_i will lead to operation in the split stability region; defined by Equation (10), s

τ_{ic} = lowest value of τ_i for $z = 1$ for which any value of K_c will lead to stable operation, s

ω_c = critical frequency (that is, 180 deg phase lag), s^{-1}

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Kinetics of Fixed-Bed Adsorption: A New Solution

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A new solution is obtained to the kinetics of a fixed-bed adsorber in response to a step change in feed concentration for a linear equilibrium system with consideration for the resistance to mass transfer in both the mobile and stationary phases. The differential equations of continuity and mass transfer are integrated upon simulating the intraparticle concentrations with a parabola. The results agree with Rosen's rigorous but complex solution in the range of conditions of practical interest. The present solution is more convenient for computer application for the calculation of breakthrough curves.

Adsorption in packed columns is in wide industrial use for the removal or recovery of dilute components from a fluid stream. The operation is gaining in importance in processes for the removal of pollutants from wastewater and exhaust gases.

A central problem in the design of these processes is the dynamic response of the adsorption column to a step change in input. A number of investigators have studied the problem, and the solutions that have been obtained can be broadly classified into two general types: the equilibrium and the nonequilibrium theories. In the equilibrium theories, the local concentrations of the adsorbate in the mobile and stationary phases are assumed to be at equilibrium. The results have been found to describe laboratory analytical columns in which the packing particles are small, and the fluid flow rates are low. However, the equilibrium theories do not quantitatively apply to industrial adsorbers owing to significant resistance to mass transfer in both the mobile and stationary phases.

The nonequilibrium theories take into account the finite resistance to mass transfer in the mobile and stationary phases and are capable of giving a quantitative description of industrial columns. Thomas (1944) and Edeskuty and Amundson (1952) presented results obtained by considering intraparticle diffusion resistance but ignoring fluid to particle resistance. Masamune and Smith (1965) considered the finite rates of surface adsorption in conjunction with either intraparticle diffusion or external diffusion. Rosen (1952) presented an analytical solution of the combined effects of intraparticle and external diffusion for linear equilibrium systems.

A uniform temperature and pressure are considered to prevail in the column in all the analytical solutions. This condition is closely approximated in liquid adsorption systems in which the heat of adsorption is small. It is approximated in a gas adsorption system only when the feed gas is highly diluted in the adsorbate.

We present here a new solution to the same problem that Rosen addressed. In Rosen's work, the partial differential equation of Fick's law describing intraparticle diffusion is integrated in conjunction with the external mass transfer equation and the continuity equation of flow. In this work, a parabola simulates the concentration profile in the particle that is developed as a result of diffusion. Integration is then carried out with the external mass transfer equation and the continuity equation of flow.

It is a common mathematical procedure to approximate an arbitrary function with a polynomial, and examples are too many to enumerate. We mention only von Karman's (1921) expression of the velocity profile in a boundary layer in terms of a polynomial of distance from the solid surface. The solution obtained is in good agreement with Blasius' (1908) more rigorous results. For the present

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